

Peak Solutions for the fractional Nirenberg problem

Yan-Hong Chen and Youquan Zheng

ABSTRACT. In this paper, the fractional order curvature equation $(-\Delta)^\gamma u = (1 + \varepsilon K(x))u^{\frac{N+2\gamma}{N-2\gamma}}$ in \mathbb{R}^N is considered. Assuming $K(x)$ has two critical points satisfying certain local conditions, we prove the existence of two-peak solutions.

1. Introduction

On the standard sphere (\mathbb{S}^N, g_0) of dimension $N \geq 2$, the famous Nirenberg problem asks: whether there exists a function w on \mathbb{S}^N such that the scalar curvature (Gauss curvature for $N = 2$) R_g of the conformal metric $g = e^w g_0$ equals to a prescribed function K ? Equivalently, one has to solve equations

$$-\Delta_{g_0} w + 1 = K e^w, \quad \text{on } \mathbb{S}^2 \quad (1.1)$$

and

$$-\Delta_{g_0} v + \frac{N-2}{4(N-1)} R_{g_0} v = \frac{N-2}{4(N-1)} K v^{\frac{N+2}{N-2}}, \quad \text{on } \mathbb{S}^N, \quad N \geq 3, \quad (1.2)$$

where $v = e^{\frac{N-2}{4}w}$.

The linear operators appearing on the left of (1.1) and (1.2) are the well known conformal Laplacians of g_0 . Their definition can be extended to a general compact Riemannian manifold (M, g) of dimension $N \geq 2$ which we denote as P_1^g . In the 1980s, Paneitz discovered another conformally covariant operator

$$P_2^g = (-\Delta_g)^2 - \operatorname{div}_g(a_N R_g g + b_N \operatorname{Ric}_g) d + \frac{N-4}{2} Q_N^g,$$

see [39] and [19]. Here Q_N^g is the Q -curvature and Ric_g is the Ricci curvature of g , a_N, b_N are suitable constants depending on N . In [25], a sequence of conformally covariant elliptic operators $\{P_k^g\}$ was constructed for all $k \in \mathbb{N}$ if N is odd but only for $k \in \{1, \dots, N/2\}$ if N is even. In [32] and [33], the authors provided a complete characterization for fully nonlinear conformally covariant differential operators of any integer order on \mathbb{R}^N . Then the existence of conformally covariant differential operators of non-integer orders

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is a natural problem. In [41], an intrinsically defined conformally covariant pseudo-differential operator of arbitrary real number order was constructed. In [26], Graham and Zworski proved that P_k^g can be defined as the residues at $\gamma = k$ of a meromorphic family of scattering operators and thus a family of elliptic pseudodifferential operators P_γ^g for non-integer γ was given.

The fractional Laplacian $(-\Delta)^\gamma$ on \mathbb{R}^N with $\gamma \in (0, 1)$ is a classical nonlocal operator. It can be defined as

$$(-\Delta)^\gamma u = C(N, \gamma) \text{P. V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2\gamma}} dy = C(N, \gamma) \lim_{\varepsilon \rightarrow 0^+} \int_{B_\varepsilon^c(x)} \frac{u(x) - u(y)}{|x - y|^{N+2\gamma}} dy.$$

Here $C(N, \gamma) = \pi^{-(2\gamma+N/2)} \frac{\Gamma(N/2+\gamma)}{\Gamma(-\gamma)}$ and P. V. means the principal value. In the celebrated paper [7] by Caffarelli and Silvestre, the authors proved that a solution $u = u(x)$ of $(-\Delta)^\gamma u = f$ in \mathbb{R}^N can be extended to a solution $u = u(x, t)$ of the following problem

$$\begin{cases} \operatorname{div}(t^{1-2\gamma} \nabla u) = 0, & \text{in } \mathbb{R}_+^{N+1}, \\ -d_\gamma t^{1-2\gamma} \partial_t u|_{t \rightarrow 0} = f, & \text{on } \partial \mathbb{R}_+^{N+1}, \end{cases}$$

with $d_\gamma = 2^{2\gamma-1} \frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}$. In [11], Chang and Gonzalez characterized P_γ^g as a Dirichlet to Neumann operator on a conformally compact Einstein manifold by the localization method of [7].

The conformally covariant property for P_γ^g with $\gamma \in (0, \frac{N}{2})$ can be expressed as follows: if $g = v^{\frac{4}{N-2\gamma}} g_0$, then

$$P_\gamma^{g_0}(vf) = v^{\frac{N+2\gamma}{N-2\gamma}} P_\gamma^g(f) \quad (1.3)$$

holds for any smooth function f , see [11]. Similar to the formula for scalar curvature and the Paneitz-Branson Q -curvature, the Q -curvature for g of order 2γ , can be defined as

$$Q_\gamma^g = P_\gamma^g(1).$$

Thus, on a smooth compact Riemannian manifold (M, g) of dimension $N \geq 2$, one has a question: is there a representation g of the conformal class $[g]$ such that $Q_\gamma^g =$ a prescribed function K ? By (1.3), one has to solve the following semi-linear equation,

$$P_\gamma^{g_0} v = K v^{\frac{N+2\gamma}{N-2\gamma}}, \quad v > 0 \text{ on } M. \quad (1.4)$$

If $\gamma = 1$, (1.4) is the classical prescribing curvature problem. There is large mount of research on this equation, we limit ourselves to citing [13] and [29], referring to their bibliography for more works. When γ is non-integer, we refer the interested readers to [21], [23], [24], [42] and the references therein for recent progress. If (M, g) is the standard sphere (\mathbb{S}^N, g_0) , (1.4) is the fractional Nirenberg problem

$$\begin{cases} P_\gamma^{g_0} v = K v^{\frac{N+2\gamma}{N-2\gamma}} \text{ on } \mathbb{S}^N, \\ v > 0, \end{cases} \quad (1.5)$$

which was studied in [1], [29], [28] and [27]. In [1], existence of positive solutions has been proved under the Euler-Hopf-type criterion for $K(x)$ via the method of critical points

at infinity (see [4] and [5]) when $N = 2$. In [29], blow up and compactness results of solutions were obtained. In [28], existence of solutions was proved under conditions of Bahri-Coron type and a fractional Aubin inequality was also proved. In [27], a unified approach was developed for blowing up profiles, compactness and existence of solutions for (1.5). We also note that critical problems on domains of \mathbb{R}^N involving the fractional Laplacian were studied in [37] and [38].

Although the operator P_γ^g on a general manifold depends on the extension problem, the operator $P_\gamma^{g_0}$ is the 2γ order conformal Laplacian on \mathbb{S}^N and it can be uniquely expressed as

$$P_\gamma^{g_0} = \frac{\Gamma(B + \frac{1}{2} + \gamma)}{\Gamma(B + \frac{1}{2} - \gamma)}, \quad B = \sqrt{-\Delta_{g_0} + \left(\frac{N-1}{2}\right)^2}.$$

Let \mathcal{N} be the north pole of \mathbb{S}^N and

$$F : \mathbb{R}^N \rightarrow \mathbb{S}^N \setminus \{\mathcal{N}\}, \quad x \mapsto \left(\frac{2x}{1+|x|^2}, \frac{|x|^2-1}{|x|^2+1} \right)$$

be the inverse of stereographic projection operator from $\mathbb{S}^N \setminus \{\mathcal{N}\}$ to \mathbb{R}^N . Then one has the following relation

$$P_\gamma^{g_0}(\phi) \circ F = |J_F|^{-\frac{N+2\gamma}{2N}} (-\Delta)^\gamma (|J_F|^{\frac{N-2\gamma}{2N}} (\phi \circ F)) \quad \text{for all } \phi \in C^\infty(\mathbb{S}^N),$$

where $|J_F| = \left(\frac{2}{1+|x|^2}\right)^N$. Hence, for a solution v of (1.5), $u(x) = |J_F|^{\frac{n-2\gamma}{2n}} v(F(x))$ satisfies

$$(-\Delta)^\gamma u = (Q \circ F) u^{\frac{N+2\gamma}{N-2\gamma}}, \quad u > 0 \quad \text{on } \mathbb{R}^N. \quad (1.6)$$

In this paper, let $Q \circ F$ in (1.6) be of form $Q \circ F = 1 + \varepsilon K(x)$ and we are interested in peak solutions for this equation. That is to say, we consider

$$\begin{cases} (-\Delta)^\gamma u = (1 + \varepsilon K(x)) u^{\frac{N+2\gamma}{N-2\gamma}} & x \in \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \quad (1.7)$$

where $\varepsilon > 0$, $\gamma \in (0, 1)$, $2\gamma < N$ and $K \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Let S_γ be the best constant of Sobolev embeddings

$$\dot{H}^\gamma(\mathbb{R}^N) \hookrightarrow L^{2_\gamma^*}(\mathbb{R}^N),$$

where $2_\gamma^* = 2N/(N - 2\gamma)$. So the following inequality holds

$$S_\gamma \left[\int_{\mathbb{R}^N} |u|^{2_\gamma^*} dx \right]^{\frac{2}{2_\gamma^*}} \leq \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\gamma}{2}} u|^2 dx \quad \text{for all } u \in \dot{H}^\gamma(\mathbb{R}^N). \quad (1.8)$$

From a celebrated result of Lieb in [34], one has that

$$S_\gamma = \left(2^{-2\gamma} \pi^{-\gamma} \frac{\Gamma(\frac{N-2\gamma}{2})}{\Gamma(\frac{N+2\gamma}{2})} \left[\frac{\Gamma(N)}{\Gamma(N/2)} \right]^{2\gamma/N} \right)^{-\frac{2_\gamma^*}{2}}.$$

Moreover, the extremals are of form

$$U_{x_0, \lambda}(x) = C_0 \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{N-2\gamma}{2}} \quad \forall x \in \mathbb{R}^N, \quad (1.9)$$

where $C_0 \in \mathbb{R} \setminus \{0\}$, $\lambda > 0$ are constants and $x_0 \in \mathbb{R}^N$ is a fixed point. In [14] and [31], the authors proved that all positive solutions of the Euler-Lagrange equation of the best constant problem for (1.8) are of form (1.9). More precisely, they proved that every positive regular solution for the semilinear partial differential equation

$$(-\Delta)^\gamma u = u^{\frac{N+2\gamma}{N-2\gamma}}$$

assumes the form

$$U_{x_0, \lambda}(x) = C_0 \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{N-2\gamma}{2}}, \quad \forall x \in \mathbb{R}^N$$

for some constant $C_0 = C_0(N, \gamma)$.

Let Σ be the set of all critical points of K (we denote such critical points as z), satisfying (after a suitable change of the coordinate system depending on z),

$$K(x) = K(z) + \sum_{i=1}^N a_i |x_i - z_i|^\beta + O(|x - z|^{\beta+\sigma})$$

for x close to z , where a_i , β and σ are some constants depending on z , $a_i \neq 0$ for $i = 1, \dots, N$, $\sum_{i=1}^N a_i < 0$, $\beta \in (1, N - 2\gamma)$ and $\sigma \in (0, 1)$. Then we have the main result of our paper.

THEOREM 1.1. *Suppose that $\gamma \in (0, 1)$, $2\gamma < N$, $K \in C^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and Σ contains at least two points. Then for each pair $z^1, z^2 \in \Sigma$, $z^1 \neq z^2$, there exists an $\varepsilon_0 > 0$ such that when $\varepsilon \in (0, \varepsilon_0)$, (1.7) has at least a solution of form*

$$u_\varepsilon = \sum_{j=1}^2 \alpha_{j,\varepsilon} U_{y_\varepsilon^j, \lambda_{j,\varepsilon}} + v_\varepsilon, \quad \text{with } \alpha_{j,\varepsilon} \rightarrow 1, y_\varepsilon^j \rightarrow z^j, \lambda_{j,\varepsilon} \rightarrow +\infty$$

and $\|v_\varepsilon\|_{\dot{H}^\gamma(\mathbb{R}^N)} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

When $\gamma = 1$, Theorem 1.1 was obtained in [8]. In [12], the author studied problem (1.7) and extended a result of [3]. Both these two papers imposed a condition relating all the critical points in Σ which implies that Σ contains only finite number of points. The condition used in Theorem 1.1 is a local one and we only assume that Σ has at least two points. If Σ contains more than two points, our result also gives a multiplicity result. Also note that, the solutions obtained in [12] have energy close to $\frac{1}{N} S_\gamma^{\frac{N}{2\gamma}}$ for small ε and those solutions need not concentrate at some point. But the corresponding energy of solutions in Theorem 1.1 is close to $\frac{2}{N} S_\gamma^{\frac{N}{2\gamma}}$ when ε is small and these solutions must concentrate at z^1 and z^2 . To prove the main theorem, we will use arguments similar to [8], see also [9], [10], [40] and [46].

The rest of this paper is organized as follows. In Section 2, we introduce some preliminary knowledge and the variational structure. In Section 3, we expand the functional and its gradient. The basic estimates needed in the proof are given in the Appendices. In Section 4, we give the proof of the Theorem 1.1.

2. Preliminaries and variational structure

Fractional Sobolev spaces have played an important role in harmonic analysis, functional analysis and partial differential equations. We refer the interested readers to [18] for an elementary introduction to this topic and a wide list of related references. Corresponding to the fractional Sobolev spaces, fractional powers of the Laplacian operator and related nonlocal equations arise in numerous of problems both from mathematics and its concrete applications, see, for example, [2], [11], [20], [22], [30], [35], [36], [44] and [45] and the references therein.

For $\gamma \in (0, 1)$ and $N \geq 2$, the fractional Sobolev space $H^\gamma(\mathbb{R}^N)$ is defined by

$$H^\gamma(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \frac{|u(x) - u(y)|}{|x - y|^{N/2 + \gamma}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N) \right\}$$

endowed with the natural norm

$$\|u\|_{H^\gamma(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2\gamma}} dx dy \right)^{\frac{1}{2}}.$$

The so-called Gagliardo semi-norm of u is defined by

$$[u]_{H^\gamma(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N + 2\gamma}} dx dy \right)^{\frac{1}{2}}.$$

We also define $\dot{H}^\gamma(\mathbb{R}^N)$ be the homogeneous version of $H^\gamma(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ under the Gagliardo semi-norm. Let $H_0^\gamma(\mathbb{R}^N)$ denote the closure of $C_0^\infty(\mathbb{R}^N)$ under the the norm $\|\cdot\|_{H^\gamma(\mathbb{R}^N)}$. Note that, in view of Theorem 2.4 in [18], $H_0^\gamma(\mathbb{R}^N) = H^\gamma(\mathbb{R}^N)$.

Let \mathcal{S} be the Schwartz space of rapidly decaying smooth functions on \mathbb{R}^N . The topology of \mathcal{S} is generated by the seminorms

$$p_m(\varphi) = \sup_{x \in \mathbb{R}^N} (1 + |x|)^m \sum_{|\gamma| \leq m} |D^\gamma \varphi(x)|, \quad m = 0, 1, 2, \dots,$$

where $\varphi \in \mathcal{S}$. Denote the topological dual of \mathcal{S} by \mathcal{S}' . For any $\varphi \in \mathcal{S}$,

$$\mathcal{F}\varphi(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi \cdot x} \varphi(x) dx$$

is the usual Fourier transformation of φ and one can extend \mathcal{F} from \mathcal{S} to \mathcal{S}' . The relation between the fractional operator $(-\Delta)^\gamma$ and the fractional Sobolev space $H^\gamma(\mathbb{R}^N)$ is yield in the following identity,

$$[u]_\gamma = C \left(\int_{\mathbb{R}^N} |\xi|^{2\gamma} |\mathcal{F}u(\xi)|^2 d\xi \right)^{\frac{1}{2}} = C \|(-\Delta)^{\frac{\gamma}{2}} u\|_{L^2(\mathbb{R}^N)}$$

for a suitable positive constant $C = C(N, \gamma)$.

For $y = (y^1, y^2) \in \mathbb{R}^N \times \mathbb{R}^N$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$, we define the space

$$E_{y,\lambda}^2 = \{v \in \dot{H}^\gamma(\mathbb{R}^N) \mid \langle U_{y^j, \lambda_j}, v \rangle = \langle \frac{\partial U_{y^j, \lambda_j}}{\partial \lambda_j}, v \rangle = \langle \frac{\partial U_{y^j, \lambda_j}}{\partial y_i^j}, v \rangle = 0, \\ \text{for } j = 1, \dots, 2, \quad i = 1, \dots, N\}.$$

Our aim is to find two-peak solutions u_ε for (1.7) of form

$$u_\varepsilon = \alpha_{1,\varepsilon} U_{y_\varepsilon^1, \lambda_{1,\varepsilon}} + \alpha_{2,\varepsilon} U_{y_\varepsilon^2, \lambda_{2,\varepsilon}} + v_\varepsilon,$$

with $\alpha_\varepsilon = (\alpha_{1,\varepsilon}, \alpha_{2,\varepsilon}) \rightarrow (1, 1)$, $y_\varepsilon = (y_\varepsilon^1, y_\varepsilon^2) \rightarrow (z^1, z^2)$, $\lambda_{j,\varepsilon} \rightarrow +\infty$ for $j = 1, 2$ and $v_\varepsilon \in E_{y_\varepsilon, \lambda_\varepsilon}^2$ satisfying $\|v_\varepsilon\|_{\dot{H}^\gamma(\mathbb{R}^N)} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

For each $z^1, z^2 \in \Sigma$, $z^1 \neq z^2$, $\mu > 0$, let us define

$$D_\mu = \{(y, \lambda) \mid y = (y^1, y^2) \in \overline{B_\mu(z^1)} \times \overline{B_\mu(z^2)}, \lambda = (\lambda_1, \lambda_2) \in (\frac{1}{\mu}, +\infty) \times (\frac{1}{\mu}, +\infty)\}$$

and

$$M_\mu = \{(\alpha, y, \lambda, v) \mid \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+, (y, \lambda) \in D_\mu, v \in E_{y,\lambda}^2, \\ |\alpha_1 - 1| \leq \mu, |\alpha_2 - 1| \leq \mu, \|v\| \leq \mu\}.$$

Furthermore, define

$$I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\gamma}{2}} u(x)|^2 dx - \frac{N-2\gamma}{2N} \int_{\mathbb{R}^N} (1 + \varepsilon K) |u|^{\frac{2N}{N-2\gamma}} dx \quad (2.1)$$

and

$$J_\varepsilon(\alpha, y, \lambda, v) = I_\varepsilon\left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v\right).$$

It is well known that, when $\mu > 0$ is sufficiently small, $(\alpha, y, \lambda, v) \in M_\mu$ is a critical point of J_ε in M_ε if and only if $u = \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v$ is a critical point of I_ε in $\dot{H}^\gamma(\mathbb{R}^N)$. Moreover, solutions of form $u = \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v$ for (1.7) are positive. For completeness, we will give the proof of these two facts in Lemma A.5 and Lemma A.6. In order to prove the main theorem, we only need to find a critical point of J_ε in M_ε .

From the Lagrange multiplier theorem, $(\alpha, y, \lambda, v) \in M_\mu$ is a critical point of J_ε in M_ε if and only if the following equations are satisfied,

$$\frac{\partial J_\varepsilon}{\partial \alpha_j} = 0, \quad j = 1, 2, \quad (2.2)$$

$$\frac{\partial J_\varepsilon}{\partial v} = \sum_{j=1}^2 A_j U_{y^j, \lambda_j} + \sum_{j=1}^2 B_j \frac{\partial U_{y^j, \lambda_j}}{\partial \lambda_j} + \sum_{j=1}^2 \sum_{i=1}^N C_{ji} \frac{\partial U_{y^j, \lambda_j}}{\partial y_i^j}, \quad (2.3)$$

$$\frac{\partial J_\varepsilon}{\partial y_i^j} = B_j \langle \frac{\partial^2 U_{y^j, \lambda_j}}{\partial \lambda_j \partial y_i^j}, v \rangle + \sum_{l=1}^N C_{jl} \langle \frac{\partial^2 U_{y^j, \lambda_j}}{\partial y_l^j \partial y_i^j}, v \rangle, \quad i = 1, \dots, N, \quad j = 1, 2, \quad (2.4)$$

$$\frac{\partial J_\varepsilon}{\partial \lambda_j} = B_j \left\langle \frac{\partial^2 U_{y^j, \lambda_j}}{\partial \lambda_j^2}, v \right\rangle + \sum_{l=1}^N C_{jl} \left\langle \frac{\partial^2 U_{y^j, \lambda_j}}{\partial y_l^j \partial \lambda_j}, v \right\rangle, \quad j = 1, 2, \quad (2.5)$$

for some $A_j, B_j, C_{ji} \in \mathbb{R}, j = 1, 2, i = 1, \dots, N$.

3. Expansion of J_ε and its gradient

Based on the estimates in Appendices B and C, we will give the expansion of J_ε and its gradient. Denote $\varepsilon_{12} = \frac{1}{(\lambda_1 \lambda_2)^{\frac{N-2\gamma}{2}}}$.

For $z^1, z^2 \in \Sigma$ and $(y, \lambda) \in D_\mu$. Let $\hat{\alpha} = ((1 + \varepsilon K(z^1))^{-\frac{N-2\gamma}{4}}, (1 + \varepsilon K(z^2))^{-\frac{N-2\gamma}{4}})$, $\bar{\alpha} = \alpha - \hat{\alpha}, \omega = (\bar{\alpha}, v) \in \mathbb{R}^2 \times E_{y, \lambda}^2$. Then $\hat{J}_\varepsilon(y, \lambda, \omega) := J_\varepsilon(\alpha, y, \lambda, v)$ can be expanded by

$$\hat{J}_\varepsilon(y, \lambda, \omega) = \hat{J}_\varepsilon(y, \lambda, 0) + \langle f_\varepsilon, \omega \rangle + \frac{1}{2} \langle Q_\varepsilon \omega, \omega \rangle + R_\varepsilon(\omega).$$

Here $f_\varepsilon \in \mathbb{R}^2 \times E_{y, \lambda}^2$ is a functional given by

$$\begin{aligned} \langle f_\varepsilon, \omega \rangle = & - \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 1} v dx + \sum_{k=1}^2 \bar{\alpha}_k \\ & \left(\left\langle \sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j}, U_{y^k, \lambda_k} \right\rangle - \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 1} U_{y^k, \lambda_k} dx \right), \end{aligned}$$

Q_ε is a quadratic form on $\mathbb{R}^2 \times E_{y, \lambda}^2$ defined as

$$\begin{aligned} & \langle Q_\varepsilon \omega, \omega \rangle \\ = & \sum_{k=1}^2 \bar{\alpha}_k \bar{\alpha}_l \left(\langle U_{y^l, \lambda_l}, U_{y^k, \lambda_k} \rangle - (2_\gamma^* - 1) \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} U_{y^l, \lambda_l} U_{y^k, \lambda_k} dx \right) \\ & + \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}^2 - (2_\gamma^* - 1) \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} v^2 dx \\ & - 2(2_\gamma^* - 1) \sum_{k=1}^2 \bar{\alpha}_k \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} U_{y^k, \lambda_k} v dx \end{aligned}$$

and R_ε is the remainder term satisfying

$$D^{(i)} R_\varepsilon(\omega) = O(\|\omega\|_{\dot{H}^\gamma(\mathbb{R}^N)}^{2+\theta-i}) \quad i = 0, 1, 2$$

for some constant $\theta > 0$.

From Lemmas B.1 and B.2 in Appendix B, if $\mu > 0, \varepsilon > 0$ are small enough, then for each $(y, \lambda) \in D_\mu$, Q_ε is invertible. Moreover, there exists $C > 0$, which is independent of

(y, λ) , such that $\|Q_\varepsilon^{-1}\|_{\dot{H}^\gamma(\mathbb{R}^N)} \leq C$. From Lemmas C.2, C.3 and C.4 in Appendix C, we have

$$|\langle f_\varepsilon, \omega \rangle| = O \left(\varepsilon \sum_{j=1}^2 \left(|y^j - z^j|^{\beta_j} + \frac{1}{\lambda_j^{\beta_j}} \right) + \varepsilon_{12}^{\frac{1}{2}+\tau} \right) \|\omega\|_{\dot{H}^\gamma(\mathbb{R}^N)},$$

where $\tau > 0$ is some constant.

Consider the equation $D\hat{J}_\varepsilon(y, \lambda, \omega) = 0$, where D is the derivative with respect to the variable ω . It is equivalent to

$$f_\varepsilon + Q_\varepsilon \omega + DR(\omega) = 0. \quad (3.1)$$

By the implicit function theorem, there exists a unique C^1 map $w(y, \lambda)$ satisfying (3.1) and

$$\|\omega\|_{\dot{H}^\gamma(\mathbb{R}^N)} \leq C \|f_\varepsilon\|_{\dot{H}^\gamma(\mathbb{R}^N)}.$$

Thus we obtain $\omega(y, \lambda) \in \mathbb{R}^2 \times E_{y,\lambda}^2$ such that $D\hat{J}_\varepsilon(y, \lambda, \omega) = 0$ for each fixed $(y, \lambda) \in D_\mu$. So we have the following

LEMMA 3.1. *Suppose $z^1, z^2 \in \Sigma$ and $(y, \lambda) \in D_\mu$. Then there exist $\varepsilon_0 > 0$ and $\mu_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0)$ and $\mu \in (0, \mu_0)$, there is a unique C^1 map $(y, \lambda) \in D_\mu \rightarrow (\alpha_\varepsilon(y, \lambda), v_\varepsilon(y, \lambda)) \in \mathbb{R}^2 \times \dot{H}^\gamma(\mathbb{R}^N)$ such that $v_\varepsilon \in E_{y,\lambda}^2$, $(\alpha_\varepsilon(y, \lambda), y, \lambda, v_\varepsilon(y, \lambda))$ satisfies (2.2) and (2.3), $\alpha_\varepsilon = (\alpha_{\varepsilon,1}, \alpha_{\varepsilon,2})$ and v_ε satisfy the following estimate,*

$$\sum_{j=1}^2 \left| \alpha_{j,\varepsilon} - (1 + \varepsilon K(z^j))^{-\frac{N-2\gamma}{4}} \right| + \|v_\varepsilon\|_{\dot{H}^\gamma(\mathbb{R}^N)} = O \left(\varepsilon \sum_{j=1}^2 \left(|y^j - z^j|^{\beta_j} + \frac{1}{\lambda_j^{\beta_j}} \right) + \varepsilon_{12}^{\frac{1}{2}+\tau} \right)$$

as $\varepsilon \rightarrow 0$, where $\tau > 0$ is some constant.

LEMMA 3.2. *Let $(y, \lambda) \in D_\mu$, (α, v) be obtained as in Lemma 3.1. Then for $\mu > 0$ and $\varepsilon > 0$ small enough, it holds that*

$$\begin{aligned} \frac{\partial J_\varepsilon(\alpha, y, \lambda, v)}{\partial \lambda_k} &= C_{N,\beta_k} \frac{\varepsilon}{\lambda_k^{\beta_k+1}} \sum_{i=1}^N a_i^k + \frac{C_0 \varepsilon_{12}}{\lambda_k |z^1 - z^2|^{N-2\gamma}} + O \left(\frac{\varepsilon \varepsilon_{12}}{\lambda_k} \right) + O \left(\frac{\varepsilon_{12}^{1+\tau_1}}{\lambda_k} \right) \\ &\quad + O \left(\frac{\varepsilon}{\lambda_k^{\beta_k}} |y^k - z^k| \right) + O \left(\frac{\varepsilon}{\lambda_k} \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j+\sigma}} + |y^j - z^j|^{\beta_j+\sigma} \right) \right), \end{aligned}$$

where $C_{N,\beta_k} > 0$ and $C_0 > 0$ are constants and $\tau_1 = \min\{\tau, \frac{1}{N}\}$.

PROOF. Without loss of generality, we assume $k = 1$. By Lemma C.7 and Lemma 3.1, we have

$$\begin{aligned}
\frac{\partial J_\varepsilon(\alpha, y, \lambda, v)}{\partial \lambda_1} &= \alpha_1 \left\langle \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v, \frac{\partial U_{y^j, \lambda_j}}{\partial \lambda_1} \right\rangle \\
&\quad - \alpha_1 \int_{\mathbb{R}^n} (1 + \varepsilon K) \left| \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v \right|^{2_\gamma^*-2} \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v \right) \frac{\partial U_{y^j, \lambda_j}}{\partial \lambda_1} dx \\
&= \alpha_1 \alpha_2 \int_{\mathbb{R}^n} U_{y^2, \lambda_2}^{2_\gamma^*-1} \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} dx - \alpha_1 \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^*-1} \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} dx \\
&\quad - \alpha_1 (2_\gamma^* - 1) \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^*-2} \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} v dx + O\left(\frac{1}{\lambda_1}\right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}^2 \\
&= \alpha_1 (\alpha_2 - \alpha_2^{2_\gamma^*-1}) \int_{\mathbb{R}^n} U_{y^2, \lambda_2}^{2_\gamma^*-1} \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} dx - \alpha_1^{2_\gamma^*} \varepsilon \int_{\mathbb{R}^n} K \alpha_j U_{y^1, \lambda_1}^{2_\gamma^*-1} \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} dx \\
&\quad - (2_\gamma^* - 1) \alpha_1^{2_\gamma^*-1} \alpha_2 \int_{\mathbb{R}^n} \alpha_j U_{y^1, \lambda_1}^{2_\gamma^*-2} \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} U_{y^2, \lambda_2} dx + O\left(\frac{\varepsilon \varepsilon_{12}}{\lambda_1}\right) \\
&\quad + O\left(\frac{\varepsilon_{12}^{1+\tau_1}}{\lambda_1}\right) + O\left(\frac{\varepsilon}{\lambda_1} \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j+\sigma}} + |y^j - z^j|^{\beta_j+\sigma}\right)\right) \\
&= -\varepsilon \int_{\mathbb{R}^n} K U_{y^1, \lambda_1}^{2_\gamma^*-1} \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} dx - (2_\gamma^* - 1) \int_{\mathbb{R}^n} U_{y^1, \lambda_1}^{2_\gamma^*-2} \frac{\partial U_{y^1, \lambda_1}}{\partial \lambda_1} U_{y^2, \lambda_2} dx \\
&\quad + O\left(\frac{\varepsilon \varepsilon_{12}}{\lambda_1}\right) + O\left(\frac{\varepsilon_{12}^{1+\tau_1}}{\lambda_1}\right) + O\left(\frac{\varepsilon}{\lambda_1} \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j+\sigma}} + |y^j - z^j|^{\beta_j+\sigma}\right)\right).
\end{aligned}$$

From Lemmas C.8 and C.10, we have the assertion of this lemma. \square

LEMMA 3.3. *Under the same assumptions of Lemma 3.2, it holds that*

$$\begin{aligned}
\frac{\partial J_\varepsilon(\alpha, y, \lambda, v)}{\partial y_i^k} &= -D_{N, \beta_k} a_i^k \frac{\varepsilon}{\lambda_k^{\beta_k-2}} (y_i^k - z_i^k) + O(\lambda_k \varepsilon_{12}^{1+\tau_1}) - \frac{C_2 (y_i^k - y_i^l)}{(\lambda_1 \lambda_2)^{\frac{N-2\gamma}{2}}} \\
&\quad + O\left(\frac{\varepsilon}{\lambda_k^{\beta_k-1}} \lambda_k^2 |y^k - z^k|^2\right) + O(\varepsilon \lambda_k \varepsilon_{12}) + O(\lambda_k \varepsilon_{12}^{1+\tau_1}) \\
&\quad + O\left(\lambda_k \varepsilon \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j+\hat{\sigma}}} + |y^j - z^j|^{\beta_j+\hat{\sigma}}\right)\right) + O\left(\varepsilon_{12}^{\frac{N-\gamma}{2}}\right),
\end{aligned}$$

where $D_{N, \beta_k} > 0$ and $C_2 > 0$ are constants, τ_1 and $\hat{\sigma}$ are the same as in Lemma 3.2.

PROOF. The proof is similar to Lemma 3.2. Indeed,

$$\begin{aligned}
\frac{\partial J_\varepsilon(\alpha, y, \lambda, v)}{\partial y_i^1} &= \alpha_1 \left\langle \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v, \frac{\partial U_{y^j, \lambda_j}}{\partial y_i^1} \right\rangle \\
&\quad - \alpha_1 \int_{\mathbb{R}^n} (1 + \varepsilon K) \left| \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v \right|^{2^*-2} \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v \right) \frac{\partial U_{y^j, \lambda_j}}{\partial y_i^1} dx \\
&= \alpha_1 \alpha_2 \int_{\mathbb{R}^n} U_{y^2, \lambda_2}^{2^*-1} \frac{\partial U_{y^1, \lambda_1}}{\partial y_i^1} dx - \alpha_1 \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2^*-1} \frac{\partial U_{y^1, \lambda_1}}{\partial y_i^1} dx \\
&\quad - \alpha_1 (2^* - 1) \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2^*-2} \frac{\partial U_{y^1, \lambda_1}}{\partial y_i^1} v dx + O(\lambda_1) \|v\|_{H^\gamma(\mathbb{R}^N)}^2 \\
&= \alpha_1 (\alpha_2 - \alpha_2^{2^*-1}) \int_{\mathbb{R}^n} U_{y^2, \lambda_2}^{2^*-1} \frac{\partial U_{y^1, \lambda_1}}{\partial y_i^1} dx - \alpha_1^{2^*} \varepsilon \int_{\mathbb{R}^n} K \alpha_j U_{y^1, \lambda_1}^{2^*-1} \frac{\partial U_{y^1, \lambda_1}}{\partial y_i^1} dx \\
&\quad - (2^* - 1) \alpha_1^{2^*-1} \alpha_2 \int_{\mathbb{R}^n} \alpha_j U_{y^1, \lambda_1}^{2^*-2} \frac{\partial U_{y^1, \lambda_1}}{\partial y_i^1} U_{y^2, \lambda_2} dx + O(\varepsilon \lambda_1 \varepsilon_{12}) \\
&\quad + O(\lambda_1 \varepsilon_{12}^{1+\tau_1}) + O\left(\lambda_1 \varepsilon \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j+\sigma}} + |y^j - z^j|^{\beta_j+\sigma} \right)\right) \\
&= -\varepsilon \int_{\mathbb{R}^n} K U_{y^1, \lambda_1}^{2^*-1} \frac{\partial U_{y^1, \lambda_1}}{\partial y_i^1} dx - (2^* - 1) \int_{\mathbb{R}^n} U_{y^1, \lambda_1}^{2^*-2} \frac{\partial U_{y^1, \lambda_1}}{\partial y_i^1} U_{y^2, \lambda_2} dx \\
&\quad + O(\varepsilon \lambda_1 \varepsilon_{12}) + O(\lambda_1 \varepsilon_{12}^{1+\tau_1}) + O\left(\lambda_1 \varepsilon \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j+\sigma}} + |y^j - z^j|^{\beta_j+\sigma} \right)\right).
\end{aligned}$$

By Lemmas C.9 and C.11, we have the conclusion. This completes the proof. \square

LEMMA 3.4. For $(y, \lambda) \in D_\mu$, let $(\alpha, v) \in \mathbb{R}^2 \times E_{y, \lambda}^2$ be obtained in Lemma 3.1. Then the following estimates hold

$$\begin{aligned}
B_k &= O(\lambda_k \varepsilon_{12}) + O\left(\lambda_k^2 \sum_{j=1}^2 \left(\frac{\varepsilon}{\lambda_j^{\beta_j+1}} + \varepsilon |y^j - z^j|^{\beta_j+1} \right)\right), \\
C_{ki} &= O\left(\frac{1}{\lambda_k^2} \left(\frac{\varepsilon}{\lambda_k^{\beta_k-1}} \lambda_k |y^k - z^k| + \varepsilon_{12} \right)\right) + O\left(\frac{1}{\lambda_k} \sum_{j=1}^2 \left(\frac{\varepsilon}{\lambda_j^{\beta_j-1+\sigma}} + \varepsilon \lambda_j |y^j - z^j|^{\beta_j+\sigma} \right)\right),
\end{aligned}$$

here, $\sigma > 0$ is some constant.

PROOF. For $\varphi \in \dot{H}^\gamma(\mathbb{R}^N)$, we have

$$\left\langle \frac{\partial J_\varepsilon}{\partial v}, \varphi \right\rangle = \sum_{j=1}^2 A_j \langle U_{y^j, \lambda_j}, \varphi \rangle + \sum_{j=1}^2 B_j \left\langle \frac{\partial U_{y^j, \lambda_j}}{\partial \lambda_j}, \varphi \right\rangle + \sum_{j=1}^2 \sum_{i=1}^N C_{ij} \left\langle \frac{\partial U_{y^j, \lambda_j}}{\partial y_i^j}, \varphi \right\rangle.$$

Choosing $\varphi = U_{y^k, \lambda_k}$, $\frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k}$, $\frac{\partial U_{y^k, \lambda_k}}{\partial y_h^k}$, $k = 1, 2$, $h = 1, \dots, N$ respectively and using the fact that

$$\left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k} \right\rangle = \frac{\partial J_\varepsilon}{\partial \lambda_k}, \quad \left\langle \frac{\partial J_\varepsilon}{\partial v}, \frac{\partial U_{y^k, \lambda_k}}{\partial y_h^k} \right\rangle = \frac{\partial J_\varepsilon}{\partial y_h^k},$$

we obtain a system of equations of A_j , B_j , C_{ji} . Moreover, the coefficient matrix is quasi diagonal. Using the estimate in Lemmas 3.2, 3.3 and in the estimates in appendix C, we obtain the desired estimates. \square

4. Proof of the main result.

Inspired by [8], we set $L_\varepsilon = \varepsilon^{-\frac{\beta_1 \beta_2}{\frac{N-2\gamma}{2}(\beta_1 + \beta_2) - \beta_1 \beta_2}}$. By Lemma 3.1, we only need to show that equations (2.4) and (2.5) are satisfied by some $(y, \lambda) \in D_\mu$.

By Lemmas 3.2-3.4, (2.4) and (2.5) are equivalent to the following system,

$$\frac{\varepsilon}{\lambda_k^{\beta_k}} \lambda_k (y_i^k - z_i^k) = O \left(\varepsilon \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j + \sigma}} + |y^j - z^j|^{\beta_j + \sigma} \right) \right) + O \left(\frac{\varepsilon_{12}}{\lambda_k} \right), \quad k = 1, 2, \quad i = 1, \dots, N, \quad (4.1)$$

$$\sum_{i=1}^N a_i^k \frac{\varepsilon}{\lambda_k^{\beta_k}} + \frac{d_k}{(\lambda_1 \lambda_2)^{\frac{N-2\gamma}{2}}} = O(\varepsilon \varepsilon_{12}) + O(\varepsilon_{12}^{1+\tau_1}) + O \left(\varepsilon \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j + \sigma}} + |y^j - z^j|^{\beta_j + \sigma} \right) \right), \quad (4.2)$$

where $d_k > 0$, $\tau_1 > 0$ are constants.

Let

$$\lambda_1 = t_1 L_\varepsilon^{\beta_1^{-1}}, \quad \lambda_2 = t_2 L_\varepsilon^{\beta_2^{-1}}, \quad t_1, t_2 \in [\gamma_1, \gamma_2],$$

$$y^1 - z^1 = \lambda_1^{-1} x^1, \quad y^2 - z^2 = \lambda_2^{-1} x^2, \quad x^1, x^2 \in B_\delta(0).$$

Then (4.1) and (4.2) are equivalent to

$$x^k = o_\varepsilon(1) \quad k = 1, 2, \quad (4.3)$$

$$t_k^{-\beta_k} + \left(\sum_{i=1}^N a_i^k \right)^{-1} d_k (t_1 t_2)^{-\frac{N-2\gamma}{2}} = o_\varepsilon(1), \quad k = 1, 2. \quad (4.4)$$

Let

$$f(x^1, x^2) = (x^1, x^2), \quad (x^1, x^2) \in \Omega_1 := B_\delta(0) \times B_\delta(0),$$

$$g(t_1, t_2) = (g_1(t_1, t_2), g_2(t_1, t_2)), \quad (t_1, t_2) \in \Omega_2 := [\gamma_1, \gamma_2] \times [\gamma_1, \gamma_2],$$

$$g_k(t_1, t_2) = \frac{1}{t_k^{\beta_k}} - \frac{m_k}{(t_1 t_2)^{\frac{N-2\gamma}{2}}}, \quad \text{where } m_k = -\frac{d_k}{\sum_{i=1}^N a_i^k} > 0.$$

Then

$$\deg(f, \Omega_1, 0) = 1. \quad (4.5)$$

It is easy to see that $g = 0$ has a unique solution (t_1^*, t_2^*) in $[\gamma_1, \gamma_2] \times [\gamma_1, \gamma_2]$ when γ_1 is small and $\gamma_2 > 0$ is large enough. Moreover, it holds that,

$$\begin{aligned} \frac{\partial g_1}{\partial t_1} \Big|_{(t_1^*, t_2^*)} &= \frac{1}{t_1^*} \left(-\beta_1 + \frac{N-2\gamma}{2} \right) \frac{m_1}{(t_1^* t_2^*)^{\frac{N-2\gamma}{2}}}, \\ \frac{\partial g_1}{\partial t_2} \Big|_{(t_1^*, t_2^*)} &= \frac{1}{t_2^*} \frac{(N-2\gamma)m_1}{2(t_1^* t_2^*)^{\frac{N-2\gamma}{2}}}, \\ \frac{\partial g_2}{\partial t_1} \Big|_{(t_1^*, t_2^*)} &= \frac{1}{t_1^*} \frac{(N-2\gamma)m_2}{2(t_1^* t_2^*)^{\frac{N-2\gamma}{2}}}, \\ \frac{\partial g_2}{\partial t_2} \Big|_{(t_1^*, t_2^*)} &= \frac{1}{t_2^*} \left(-\beta_2 + \frac{N-2\gamma}{2} \right) \frac{m_2}{(t_1^* t_2^*)^{\frac{N-2\gamma}{2}}}. \end{aligned}$$

Thus the Jacobian determinant of g at (t_1^*, t_2^*) satisfies

$$J_g \Big|_{(t_1^*, t_2^*)} = (\beta_1 \beta_2 - (\beta_1 + \beta_2) \frac{N-2\gamma}{2}) \frac{m_1 m_2}{(t_1^* t_2^*)^{N-\gamma}} < 0.$$

So

$$\deg(g, \Omega_2, 0) = -1. \quad (4.6)$$

From (4.5) and (4.6), we have

$$\deg((f \times g), \Omega_1 \times \Omega_2, 0) = \deg(f, \Omega_1, 0) \times \deg(g, \Omega_2, 0) = -1.$$

Hence there exists a solution for (4.3) and (4.4). This completes the proof.

Appendix A.

In this appendix, we prove that, for $\mu > 0$ sufficiently small, $(\alpha, y, \lambda, v) \in M_\mu$ is a critical point of J_ε in M_ε if and only if $u = \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v$ is a critical point of I_ε in $\dot{H}^\gamma(\mathbb{R}^N)$ and the fact that solutions of form $u = \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v$ for (1.7) are positive.

For $x \in \mathbb{R}^N$ and $\lambda > 0$, define

$$\delta(y, \lambda)(x) = U_{y, \lambda}(x) = C_0 \left(\frac{\lambda}{1 + \lambda^2 |x - y|^2} \right)^{\frac{N-2\gamma}{2}}.$$

Following the idea of [5], first, we prove

LEMMA A.1. *Let (μ_k) be a sequence of positive numbers, $\lim_{k \rightarrow +\infty} \mu_k = 0$ and let $(\alpha^k, y_k, \lambda^k), (\tilde{\alpha}^k, \tilde{y}_k, \tilde{\lambda}^k) \in B_{\mu_k} := \{(\alpha, y, \lambda) | \alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+, (y, \lambda) \in D_{\mu_k}, |\alpha_1 - 1| \leq \mu_k, |\alpha_2 - 1| \leq \mu_k\}$ be two sequences such that*

$$\lim_{k \rightarrow +\infty} \left\| \sum_{j=1}^2 \alpha_j^k U_{y_j^k, \lambda_j^k} - \sum_{j=1}^2 \tilde{\alpha}_j^k U_{\tilde{y}_j^k, \tilde{\lambda}_j^k} \right\|_{\dot{H}^\gamma(\mathbb{R}^N)} = 0. \quad (A.1)$$

Then it holds that

$$\lim_{k \rightarrow +\infty} (\lambda_j^k / \tilde{\lambda}_j^k) = 1, \quad \text{for all } j = 1, 2, \quad (A.2)$$

$$\lim_{k \rightarrow +\infty} \lambda_j^k \tilde{\lambda}_j^k |y_k^j - \tilde{y}_k^j|^2 = 0, \quad \text{for all } j = 1, 2, \quad (\text{A.3})$$

$$\lim_{k \rightarrow +\infty} |\alpha_j^k - \tilde{\alpha}_j^k| = 0, \quad \text{for all } j = 1, 2. \quad (\text{A.4})$$

PROOF. First, it holds that

$$\|\delta(y, \lambda)\|_{\dot{H}^\gamma(\mathbb{R}^N)} = 1$$

and

$$\lim_{\lambda/\lambda' + \lambda'/\lambda + \lambda\lambda'|y-y'|^2 \rightarrow +\infty} \int_{\mathbb{R}^N} (-\Delta)^{\frac{\gamma}{2}} \delta(y, \lambda) (-\Delta)^{\frac{\gamma}{2}} \delta(y', \lambda') dx = 0. \quad (\text{A.5})$$

Then by (A.1) and (A.5), we have

$$\int_{\mathbb{R}^N} |\alpha_i^k (-\Delta)^{\frac{\gamma}{2}} \delta(y_k^i, \lambda_i^k) - \tilde{\alpha}_i^k (-\Delta)^{\frac{\gamma}{2}} \delta(\tilde{y}_k^i, \tilde{\lambda}_i^k)|^2 dx = o(1) \quad \text{for } i = 1, 2. \quad (\text{A.6})$$

Hence

$$\lambda_i^k / \tilde{\lambda}_i^k + \tilde{\lambda}_i^k / \lambda_i^k + \lambda_i^k \tilde{\lambda}_i^k |y_k^i - \tilde{y}_k^i|^2 \leq c.$$

Set

$$w = C_0 \left(\frac{1}{1 + |x|^2} \right)^{\frac{N-2\gamma}{2}},$$

then from (A.6), (A.4) holds. Furthermore,

$$\begin{aligned} & \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\gamma}{2}} \omega - (-\Delta)^{\frac{\gamma}{2}} \delta(\lambda_i(y_k^i - \tilde{y}_k^i), \tilde{\lambda}_i^k / \lambda_i^k)|^2 dx \\ &= \int_{\mathbb{R}^N} |(-\Delta)^{\frac{\gamma}{2}} \delta(y_k^i, \lambda_i^k) - (-\Delta)^{\frac{\gamma}{2}} \delta(\tilde{y}_k^i, \tilde{\lambda}_i^k)|^2 dx \\ &= o(1). \end{aligned}$$

Thus we have (A.2) and (A.3). \square

LEMMA A.2. *There exists a constant $\mu_0 > 0$ such that when $\mu \in (0, \mu_0]$ and for each $u \in \dot{H}^\gamma(\mathbb{R}^N)$ satisfying*

$$\|u - \sum_{j=1}^2 U_{\bar{y}^j, \bar{\lambda}_j}\|_{\dot{H}^\gamma(\mathbb{R}^N)} \leq \mu, \quad \text{for some } (\bar{y}, \bar{\lambda}) \in D_\mu,$$

the infimum problem

$$\inf_{(\alpha, y, \lambda) \in B_{4\mu}} \|u - \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j}\|_{\dot{H}^\gamma(\mathbb{R}^N)}$$

is achieved in $B_{2\mu}$ and is not achieved in $B_{4\mu} \setminus B_{2\mu}$.

PROOF. Let us prove that the infimum cannot be achieved in $B_{4\mu_0} \setminus B_{2\mu_0}$ if μ_0 is small enough. If this is not true, then there exist $\{\mu_k\}$ with $\mu_k > 0$ and $\mu_k = o(1)$, a sequence $(\tilde{\alpha}^k, \tilde{y}_k, \tilde{\lambda}^k) \in B_{4\mu_k} \setminus B_{2\mu_k}$ and $(y_k, \lambda^k) \in D_{\mu_k}$ such that

$$\|u - \sum_{j=1}^2 U_{y_k^j, \lambda_j^k}\|_{\dot{H}^\gamma(\mathbb{R}^N)} \leq \mu_k.$$

Then we have

$$\left\| \sum_{j=1}^2 \tilde{\alpha}^k U_{\tilde{y}_k^j, \tilde{\lambda}_j^k} - \sum_{j=1}^2 U_{y_k^j, \lambda_j^k} \right\|_{\dot{H}^\gamma(\mathbb{R}^N)} = o(1).$$

By Lemma A.1, it holds that

$$\lambda_i^k / \tilde{\lambda}_i^k = 1 + o(1), \quad \text{for all } i = 1, 2, \quad (\text{A.7})$$

$$\lambda_i^k \tilde{\lambda}_i^k |y_k^i - \tilde{y}_k^i|^2 = o(1), \quad \text{for all } i = 1, 2. \quad (\text{A.8})$$

(A.7), (A.8) and the fact that $(\tilde{\alpha}^k, \tilde{y}_k, \tilde{\lambda}^k) \in B_{4\mu_k} \setminus B_{2\mu_k}$ and $(y_k, \lambda^k) \in D_{\mu_k}$ is a contradiction. The other assertion can be proved similarly. \square

LEMMA A.3. *There is a constant $\mu_0 > 0$ such that when $\mu \in (0, \mu_0]$ and for each $u \in \dot{H}^\gamma(\mathbb{R}^N)$ satisfying*

$$\|u - \sum_{j=1}^2 U_{\bar{y}^j, \bar{\lambda}_j}\|_{\dot{H}^\gamma(\mathbb{R}^N)} \leq \mu, \quad \text{for some } (\bar{y}, \bar{\lambda}) \in D_\mu,$$

the infimum problem

$$\inf_{(\alpha, y, \lambda) \in B_{4\mu}} \|u - \sum_{j=1}^2 \alpha_j U_{y_j, \lambda_j}\|_{\dot{H}^\gamma(\mathbb{R}^N)}$$

is uniquely achieved.

PROOF. By Lemma A.2, we only need to prove the uniqueness part of the assertion. We argue by contradiction. If the statement is false, then there exist $\{\mu_k\}$ with $\mu_k = o(1)$, $\{u_k\}$ such that

$$\|u_k - \sum_{j=1}^2 U_{\bar{y}_k^j, \bar{\lambda}_j^k}\|_{\dot{H}^\gamma(\mathbb{R}^N)} \leq \mu_k, \quad \text{for some } (\bar{y}_k, \bar{\lambda}^k) \in D_{\mu_k},$$

$(\alpha^k, y_k, \lambda^k), (\tilde{\alpha}^k, \tilde{y}_k, \tilde{\lambda}^k)$ in $B_{2\mu_k}$ for which the following properties hold:
first,

$$(\alpha^k, y_k, \lambda^k) \neq (\tilde{\alpha}^k, \tilde{y}_k, \tilde{\lambda}^k);$$

second, if $v^k = u^k - \sum_{j=1}^2 \alpha_j^k U_{y_k^j, \lambda_j^k}$ and $\tilde{v}^k = u^k - \sum_{j=1}^2 \tilde{\alpha}_j^k U_{\tilde{y}_k^j, \tilde{\lambda}_j^k}$, then it holds that

$$0 = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\gamma}{2}} v^k (-\Delta)^{\frac{\gamma}{2}} \delta_i^k dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\gamma}{2}} v^k (-\Delta)^{\frac{\gamma}{2}} \frac{\partial \delta_i^k}{\partial \lambda_i^k} dx \quad \text{for } i = 1, 2 \text{ and all } k, \quad (\text{A.9})$$

$$0 = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\gamma}{2}} v^k (-\Delta)^{\frac{\gamma}{2}} \frac{\partial \delta_i^k}{\partial y_i^k} dx \quad \text{for } i = 1, 2 \text{ and all } k, \quad (\text{A.10})$$

$$0 = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\gamma}{2}} \tilde{v}^k (-\Delta)^{\frac{\gamma}{2}} \tilde{\delta}_i^k dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\gamma}{2}} \tilde{v}^k (-\Delta)^{\frac{\gamma}{2}} \frac{\partial \tilde{\delta}_i^k}{\partial \tilde{\lambda}_i^k} dx \quad \text{for } i = 1, 2 \text{ and all } k, \quad (\text{A.11})$$

$$0 = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\gamma}{2}} \tilde{v}^k (-\Delta)^{\frac{\gamma}{2}} \frac{\partial \tilde{\delta}_i^k}{\partial \tilde{y}_i^k} dx \quad \text{for all } i \text{ and all } k, \quad (\text{A.12})$$

here

$$\delta_i^k = \delta(y_k^i, \lambda_i^k), \quad \tilde{\delta}_i^k = \delta(\tilde{y}_k^i, \tilde{\lambda}_i^k).$$

In the following, we shall omit the index k . Then, by Lemma A.1, we have

$$\begin{aligned} \lambda_i / \tilde{\lambda}_i &= 1 + o(1), \\ \lambda_i \tilde{\lambda}_i |x_i - \tilde{x}_i|^2 &= o(1), \\ |\alpha_i - \tilde{\alpha}_i| &= o(1). \end{aligned}$$

Also by (A.9) and (A.11), we get

$$\sum_j \int_{\mathbb{R}^N} \left(\alpha_j (-\Delta)^{\frac{\gamma}{2}} \delta_j - \tilde{\alpha}_j (-\Delta)^{\frac{\gamma}{2}} \tilde{\delta}_j \right) (-\Delta)^{\frac{\gamma}{2}} \delta_i dx = \int_{\mathbb{R}^N} (-\Delta)^{\frac{\gamma}{2}} \tilde{v} ((-\Delta)^{\frac{\gamma}{2}} \delta_i - (-\Delta)^{\frac{\gamma}{2}} \tilde{\delta}_i) dx.$$

Let $\eta_i = \tilde{\lambda}_i / \lambda_i - 1$, $a_i = \tilde{\lambda}_i (x_i - \tilde{x}_i)$, $\mu_i = \alpha_i - \tilde{\alpha}_i$. Note that $|a_i| = o(1)$, $\eta_i = o(1)$, $\mu_i = o(1)$, it is easy to see that

$$|\tilde{\delta}_j(y) - \delta_j(y)| \leq c(|\eta_j| + |a_j|) \delta_j(y),$$

$$\begin{aligned} \int_{\mathbb{R}^N} \left(\alpha_j (-\Delta)^{\frac{\gamma}{2}} \delta_j - \tilde{\alpha}_j (-\Delta)^{\frac{\gamma}{2}} \tilde{\delta}_j \right) (-\Delta)^{\frac{\gamma}{2}} \delta_i dx &= (\alpha_j - \tilde{\alpha}_j) \int_{\mathbb{R}^N} (-\Delta)^{\frac{\gamma}{2}} \delta_j (-\Delta)^{\frac{\gamma}{2}} \delta_i dx \\ &\quad + \tilde{\alpha}_j \int_{\mathbb{R}^N} \delta_i^{\frac{N+2\gamma}{N-2\gamma}} (\delta_j - \tilde{\delta}_j) dx. \end{aligned}$$

Then we have

$$\mu_i + \tilde{\alpha}_i \int_{\mathbb{R}^N} \delta_i^{\frac{N+2\gamma}{N-2\gamma}} (\delta_i - \tilde{\delta}_i) dx = o(1)(|\eta_j| + |a_j| + |\mu_j|) + o(1) \left(\int_{\mathbb{R}^N} |((-\Delta)^{\frac{\gamma}{2}} \delta_i - (-\Delta)^{\frac{\gamma}{2}} \tilde{\delta}_i)|^2 dx \right)^{\frac{1}{2}},$$

and

$$\int_{\mathbb{R}^N} \delta_i^{\frac{N+2\gamma}{N-2\gamma}} (\delta_i - \tilde{\delta}_i) dx = O(|a_i|^2 + |\eta_i|^2).$$

Hence,

$$\mu_i = o(1)(|\eta_j| + |a_j| + |\mu_j|).$$

Similarly, it holds that

$$\eta_i = o(1)(|\eta_j| + |a_j| + |\mu_j|)$$

and

$$a_i = o(1)(|\eta_j| + |a_j| + |\mu_j|).$$

Thus

$$\eta_i = 0, \quad a_i = 0, \quad \mu_i = 0 \quad \text{for all } i = 1, 2.$$

This completes the proof. \square

By Lemma A.3, we have the following direct consequence,

LEMMA A.4. *There is a constant $\mu_0 > 0$ such that when $\mu \in (0, \mu_0]$ and each $u \in \dot{H}^\gamma(\mathbb{R}^N)$ satisfying*

$$\|u - \sum_{j=1}^2 U_{\bar{y}^j, \bar{\lambda}^j}\|_{\dot{H}^\gamma(\mathbb{R}^N)} \leq \mu, \quad \text{for some } (\bar{y}, \bar{\lambda}) \in D_\mu,$$

u can be uniquely decomposed into

$$u = \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v,$$

for some $(\alpha, y, \lambda, v) \in M_\mu$.

Thus we clearly have

LEMMA A.5. *For $\mu > 0$ sufficiently small, $(\alpha, y, \lambda, v) \in M_\mu$ is a critical point of J_ε in M_ε if and only if $u = \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v$ is a critical point of I_ε in $\dot{H}^\gamma(\mathbb{R}^N)$.*

Finally, we prove

LEMMA A.6. *If $u_\varepsilon = \sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} + v$ is a critical point of (2.1), then u is positive.*

PROOF. We follow the idea of [43]. Let us write $u_\varepsilon = u_\varepsilon^+ + u_\varepsilon^-$, where $u_\varepsilon^+ = \max(0, u_\varepsilon)$ and $u_\varepsilon^- = \min(0, u_\varepsilon)$. Suppose $u_\varepsilon^- \neq 0$. Testing equation (1.7) by u_ε^- , we have

$$\int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_\varepsilon(x) - u_\varepsilon(y))(u_\varepsilon^-(x) - u_\varepsilon^-(y))}{|x - y|^{N+2\gamma}} dx dy = \int_{\mathbb{R}^N} (1 + \varepsilon K(x)) |u_\varepsilon^-|^{2^*} dx.$$

Since

$$\int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u_\varepsilon(x) - u_\varepsilon(y))(u_\varepsilon^-(x) - u_\varepsilon^-(y))}{|x - y|^{N+2\gamma}} dx dy \geq \int \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u_\varepsilon^-(x) - u_\varepsilon^-(y)|^2}{|x - y|^{N+2\gamma}} dx dy,$$

by the Sobolev embedding inequality, it holds that

$$\int_{\mathbb{R}^N} |u_\varepsilon^-|^{2^*} dx \geq c_0.$$

But

$$\int_{\mathbb{R}^N} |u_\varepsilon^-|^{2^*} dx \leq \int_{\mathbb{R}^N} |u_\varepsilon|^{2^*} dx \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

which is a contradiction. Hence $u_\varepsilon \geq 0$. By the maximum principle for the fractional Laplacian, see [6], $u_\varepsilon > 0$. This completes the proof. \square

Appendix B.

In this section, we prove the invertibility of Q_ε . We follow the idea of [15] and [16]. First, we prove that

LEMMA B.1. *Suppose $(y, \lambda) \in D_\mu$, μ is small enough, the operator defined by*

$$\langle A_{y,\lambda}v, w \rangle = \langle v, w \rangle - (2_\gamma^* - 1) \int_{\mathbb{R}^n} \sum_{j=1}^2 U_{y^j, \lambda_j}^{2_\gamma^*-2} v w dx, \quad v, w \in E_{y,\lambda}^2$$

satisfies

$$\|A_{y,\lambda}v\|_{\dot{H}^\gamma(\mathbb{R}^N)} \geq c_0 \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)} \quad \text{for } v \in E_{y,\lambda}^2.$$

.

PROOF. We argue by contradiction. Suppose that there exists $\mu_k \rightarrow 0$ and $(y_k, \lambda^k) \in D_{\mu_k}$, $v_k \in E_{y_k, \lambda^k}^2$ such that

$$\|A_{y_k, \lambda^k} v_k\|_{\dot{H}^\gamma(\mathbb{R}^N)} = o(1) \|v_k\|_{\dot{H}^\gamma(\mathbb{R}^N)}.$$

We may assume

$$\|v_k\|_{\dot{H}^\gamma(\mathbb{R}^N)} = (\lambda_1^k \lambda_2^k)^{-\frac{N}{2}}.$$

So

$$|\langle A_{y_k, \lambda^k} v_k, w \rangle| = o((\lambda_1^k \lambda_2^k)^{-\frac{N}{2}}) \|w\|_{\dot{H}^\gamma(\mathbb{R}^N)}.$$

That is,

$$\begin{aligned} \int_{\mathbb{R}^n} (-\Delta)^{\frac{\gamma}{2}} v_k (-\Delta)^{\frac{\gamma}{2}} w dx &= (2_\gamma^* - 1) \int_{\mathbb{R}^n} \sum_{j=1}^2 U_{y_k^j, \lambda_j^k}^{2_\gamma^*-2} v_k w dx \\ &= o((\lambda_1^k \lambda_2^k)^{-\frac{N}{2}}) \|w\|_{\dot{H}^\gamma(\mathbb{R}^N)}, \quad w \in E_{y_k, \lambda^k}^2. \end{aligned}$$

For each fixed i , let

$$\bar{v}_k(x) = v_k\left(\frac{1}{\lambda_i^k}x + y_k^i\right),$$

then we have

$$\begin{aligned} \int_{\mathbb{R}^n} (-\Delta)^{\frac{\gamma}{2}} \bar{v}_k (-\Delta)^{\frac{\gamma}{2}} w dx &= (2_\gamma^* - 1) \int_{\mathbb{R}^n} \frac{1}{(1 + |x|^2)^{2\gamma}} \bar{v}_k w dx \\ &= (2_\gamma^* - 1) \int_{\mathbb{R}^n} \frac{(\lambda_j^k / \lambda_i^k)^{2\gamma}}{(1 + |\frac{\lambda_j^k}{\lambda_i^k}x + \lambda_j^k(y_k^i - y_k^j)|^2)^{2\gamma}} \bar{v}_k w dx \\ &= (\lambda_i^k)^{-2\gamma} (\lambda_j^k)^{-\frac{N}{2}} o(1) \|w\|_{\dot{H}^\gamma(\mathbb{R}^N)}, \quad w \in F_{y_k, \lambda^k}^2 \end{aligned}$$

where

$$\begin{aligned} F_{y_k, \lambda^k}^2 &= \{v \in \dot{H}^\gamma(\mathbb{R}^N) \mid \langle U_{y_k, \lambda_j^k}(\frac{1}{\lambda_i^k} \cdot + y_k^i), v \rangle = \langle \frac{\partial U_{y_k, \lambda_j^k}}{\partial \lambda_j^k}(\frac{1}{\lambda_i^k} \cdot + y_k^i), v \rangle \\ &= \langle \frac{\partial U_{y_k, \lambda_j^k}}{\partial y_{ki}^j}(\frac{1}{\lambda_i^k} \cdot + y_k^i), v \rangle = 0, \text{ for } j = 1, 2, \quad i = 1, \dots, N\}. \end{aligned}$$

Since $\|\bar{v}_k\|_{\dot{H}^\gamma(\mathbb{R}^N)} = (\lambda_i^k)^{-2\gamma}(\lambda_j^k)^{-\frac{N}{2}}$, we may conclude that

$$\bar{v}_k \rightharpoonup v \quad \text{weakly in } \dot{H}^\gamma(\mathbb{R}^N),$$

$$\bar{v}_k \rightarrow v \quad \text{strongly in } L^p(\mathbb{R}^N) \text{ with } p \in [2, 2_\gamma^*).$$

Moreover, it is easy to see that v satisfies

$$\langle U, v \rangle = \langle \frac{\partial U}{\partial x_j}, v \rangle = 0.$$

Now we claim that $v = 0$. Assume this for the moment. Since for each $L > 0$, we have

$$\begin{aligned} \sum_{j=1}^2 \int_{\mathbb{R}^n} U_{y_k, \lambda_j^k}^{2_\gamma^*-2} v_k^2 dx &= \sum_{j=1}^2 \int_{B_{L \frac{1}{\lambda_i^k}}(y_k^i)} U_{y_k, \lambda_j^k}^{2_\gamma^*-2} v_k^2 dx + (\frac{1}{L^{2N-1}})^{\frac{4\gamma}{N+2\gamma}} (\lambda_1^k \lambda_2^k)^{-N} \\ &= \left((\frac{1}{\lambda_1^k})^{-N} + (\frac{1}{\lambda_1^k})^{-N} \right) o(1) + (\frac{1}{L^{2N-1}})^{\frac{4\gamma}{N+2\gamma}} (\lambda_1^k \lambda_2^k)^{-N} \\ &= (\lambda_1^k \lambda_2^k)^{-N} (o(1) + (\frac{1}{L^{2N-1}})^{\frac{4\gamma}{N+2\gamma}}). \end{aligned}$$

Then we have

$$\|v_k\|_{\dot{H}^\gamma(\mathbb{R}^N)}^2 = (\lambda_1^k \lambda_2^k)^{-N} (o(1) + (\frac{1}{L^{2N-1}})^{\frac{4\gamma}{N+2\gamma}}).$$

This is a contradiction.

So it remains to prove that $v = 0$. First, we claim that v satisfies

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\gamma}{2}} v (-\Delta)^{\frac{\gamma}{2}} w dx - (2_\gamma^* - 1) \int_{\mathbb{R}^n} U^{2_\gamma^*-2} v w dx = 0, \quad w \in F, \quad (\text{B.1})$$

where

$$F = \{w \in \dot{H}^\gamma(\mathbb{R}^N) \mid \langle U, w \rangle = \langle \frac{\partial U}{\partial x_j}, w \rangle = 0, \quad j = 1, 2, \dots, N\}.$$

Indeed, for each $w \in F$, we can choose $\alpha_j^{(k)}$, $\beta_j^{(k)}$ and $\gamma_{jl}^{(k)}$ such that

$$\begin{aligned} \eta_k &= w - \sum_{j=1}^2 \alpha_j^{(k)} U_{y_k, \lambda_j^k}(\frac{1}{\lambda_i^k} \cdot + y_k^i) - \sum_{j=1}^2 \beta_j^{(k)} \frac{\partial U_{y_k, \lambda_j^k}}{\partial \lambda_j^k}(\frac{1}{\lambda_i^k} \cdot + y_k^i) \\ &\quad - \sum_{j=1}^2 \sum_{l=1}^2 \gamma_{jl}^{(k)} \frac{\partial U_{y_k, \lambda_j^k}}{\partial y_{kl}^j}(\frac{1}{\lambda_i^k} \cdot + y_k^i) \in F_{y_k, \lambda^k}^2. \end{aligned}$$

And it is easy to see that $\alpha_j^{(k)} \rightarrow 0$, $\beta_j^{(k)} \rightarrow 0$ and $\gamma_{jl}^{(k)} \rightarrow 0$ as $k \rightarrow \infty$. Hence we have (B.1). Then from the non-degenerate result of [17], $v = 0$. This completes the proof. \square

LEMMA B.2. *Suppose $(y, \lambda) \in D_\mu$, μ and ε small enough, then there exists $\delta > 0$ such that for $v \in E_{y, \lambda}^2$ we have*

$$\|v\|_{H^\gamma(\mathbb{R}^N)}^2 - (2_\gamma^* - 1) \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \hat{\alpha} U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} v^2 dx \geq \delta \|v\|_{H^\gamma(\mathbb{R}^N)}^2.$$

The proof of this lemma is the same as Lemma A.4 in [8], so we omit it.

Appendix C.

In this appendix, we prove some estimates needed in the proof of our main results.

LEMMA C.1. *Let $\alpha, \beta > 1$ such that $\alpha + \beta = 2_\gamma^*$, then there exists $\theta > 0$ such that*

$$\begin{aligned} \int_{\mathbb{R}^N} U_{y^i, \lambda_i}^{\frac{N+2}{N-2}} U_{y^j, \lambda_j} dx &= C_0^{\frac{2N}{N-2\gamma}} C_1 \varepsilon_{ij} + O(\varepsilon_{ij}^{\frac{N}{N-2\gamma}}), \\ \int_{\mathbb{R}^N} U_{y^i, \lambda_i}^{\frac{N}{N-2\gamma}} U_{y^j, \lambda_j}^{\frac{N}{N-2\gamma}} dx &= O(\varepsilon_{ij}^{\frac{N}{N-2\gamma}} \log \varepsilon_{ij}^{-1}), \\ \int_{\mathbb{R}^N} U_{y^i, \lambda_i}^\alpha U_{y^j, \lambda_j}^\beta dx &= O(\varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{\frac{N-2\gamma}{N} \theta}) \quad \text{with } \theta = \inf(\alpha, \beta), \\ \left| \frac{\partial \delta_i}{\partial x_i} \right| &\leq C \lambda_i, \\ \left| \frac{\partial \delta_i}{\partial \lambda_i} \right| &\leq \frac{C}{\lambda_i}. \end{aligned}$$

PROOF. The proof is very similar to that of [4] and [43], so we only prove the first estimate. Set

$$\begin{aligned} \delta_i(x) &= C_0 \left(\frac{\lambda_i}{1 + \lambda_i^2 |x - x_i|^2} \right)^{\frac{N-2\gamma}{2}}, \\ I &= \int_{\mathbb{R}^N} \delta_i^{\frac{N+2\gamma}{N-2\gamma}} \delta_j dx. \end{aligned}$$

Then it holds that

$$\begin{aligned} I &= C_0^{\frac{2N}{N-2\gamma}} \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^{\frac{N+2\gamma}{2}} \left(\frac{\lambda_j}{\lambda_i} + \left| \sqrt{\frac{\lambda_i}{\lambda_j}} - \sqrt{\lambda_i \lambda_j} d_{ij} \right|^2 \right)^{\frac{N-2\gamma}{2}}} dx \\ &= C_0^{\frac{2N}{N-2\gamma}} \int_{\mathbb{R}^N} \frac{1}{(1 + |x|^2)^{\frac{N+2\gamma}{2}} \left(\frac{\lambda_i}{\lambda_j} + \left| \sqrt{\frac{\lambda_j}{\lambda_i}} + \sqrt{\lambda_i \lambda_j} d_{ij} \right|^2 \right)^{\frac{N-2\gamma}{2}}} dx. \end{aligned}$$

First, we assume that

$$\mu = \max\{\lambda_i/\lambda_j, \lambda_j/\lambda_i, \lambda_i \lambda_j |x_i - x_j|^2\} = \lambda_i/\lambda_j.$$

By Taylor expansion, it holds that

$$\begin{aligned} \frac{\lambda_i}{\lambda_j} + \left| \sqrt{\frac{\lambda_j}{\lambda_i}} + \sqrt{\lambda_i \lambda_j} d_{ij} \right|^2 &= \left(\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |d_{ij}|^2 \right) \times \left\{ 1 + \frac{\frac{\lambda_j}{\lambda_i} |x|^2 + 2\lambda_j x \cdot d_{ij}}{\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |d_{ij}|^2} \right\} \\ &\left\{ \frac{1}{\frac{\lambda_i}{\lambda_j} + \left| \sqrt{\frac{\lambda_j}{\lambda_i}} + \sqrt{\lambda_i \lambda_j} d_{ij} \right|^2} \right\}^{\frac{N-2\gamma}{2}} = \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |d_{ij}|^2} \right)^{\frac{N-2\gamma}{2}} \\ &\times \left\{ 1 - (N-2\gamma) \frac{\lambda_j x \cdot d_{ij}}{\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |d_{ij}|^2} + O\left(\frac{1}{\lambda_i \lambda_j |d_{ij}|^2} |x|^2 \right) \right\}. \end{aligned}$$

Then we have

$$\begin{aligned} &\int_{|x| \leq \frac{\sqrt{\mu}}{10}} \frac{1}{(1+|x|^2)^{\frac{N+2\gamma}{2}} \left(\frac{\lambda_i}{\lambda_j} + \left| \sqrt{\frac{\lambda_j}{\lambda_i}} + \sqrt{\lambda_i \lambda_j} d_{ij} \right|^2 \right)^{\frac{N-2\gamma}{2}}} dx \\ &= \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |d_{ij}|^2} \right)^{\frac{N-2\gamma}{2}} \left(C_1 + O\left(\frac{1}{\mu^\gamma} \right) \right). \end{aligned}$$

Moreover, by easy computations, we have

$$\begin{aligned} &\int_{|x| \geq \frac{\sqrt{\mu}}{10}} \frac{1}{(1+|x|^2)^{\frac{N+2\gamma}{2}}} dx = O\left(\left(\frac{1}{\mu} \right)^\gamma \right), \\ &\int_{|x| \leq \frac{\sqrt{\mu}}{10}} \frac{x}{(1+|x|^2)^{\frac{N+2\gamma}{2}}} dx = 0, \\ &\int_{|x| \leq \frac{\sqrt{\mu}}{10}} \frac{1}{(1+|x|^2)^{\frac{N+2\gamma}{2}}} dx = C_1 + O\left(\left(\frac{1}{\mu} \right)^\gamma \right), \\ &\int_{|x| \leq \frac{\sqrt{\mu}}{10}} \frac{x^2}{(1+|x|^2)^{\frac{N+2\gamma}{2}}} dx = \mu^{1-\gamma}. \end{aligned}$$

Hence it holds that

$$\begin{aligned} &\int_{|x| \leq \frac{\sqrt{\mu}}{10}} \frac{1}{(1+|x|^2)^{\frac{N+2\gamma}{2}} \left(\frac{\lambda_i}{\lambda_j} + \left| \sqrt{\frac{\lambda_j}{\lambda_i}} + \sqrt{\lambda_i \lambda_j} d_{ij} \right|^2 \right)^{\frac{N-2\gamma}{2}}} dx \\ &= \left(\frac{1}{\frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j |d_{ij}|^2} \right)^{\frac{N-2\gamma}{2}} \left(C_1 + O\left(\frac{1}{\mu^\gamma} \right) \right) \\ &= C_1 \varepsilon_{ij} + O(\varepsilon_{ij}^{\frac{N}{N-2\gamma}}). \end{aligned}$$

For the case $\mu = \lambda_j/\lambda_i$, the proof is similar. So we are left with the case

$$\mu = \lambda_i \lambda_j |x_i - x_j|^2.$$

Let

$$B_1 = \{x \in \mathbb{R}^N \mid |x + \lambda_i d_{ij}| \leq 1/10 \lambda_i |d_{ij}|\},$$

$$B_2 = \{x \in \mathbb{R}^N \mid |x| \leq \frac{\sqrt{\mu}}{10}\}$$

and

$$L(x) = \frac{1}{(1 + |x|^2)^{\frac{N+2\gamma}{2}} \left(\frac{\lambda_i}{\lambda_j} + \left| \sqrt{\frac{\lambda_j}{\lambda_i}} x + \sqrt{\lambda_i \lambda_j} d_{ij} \right|^2 \right)^{\frac{N-2\gamma}{2}}}.$$

Then

$$\int_{(B_1 \cup B_2)^c} L(x) dx \leq \frac{C}{\mu^{\frac{N-2\gamma}{2}}} \int_{\sqrt{\mu}}^{+\infty} \frac{r^{N-1} dr}{(1 + r^2)^{\frac{N+2\gamma}{2}}} = O(\varepsilon_{ij}^{\frac{N}{N-2\gamma}}).$$

On B_1 , we have

$$|x| \geq 9/10 \lambda_i |d_{ij}|.$$

Thus

$$\begin{aligned} \int_{B_1} L(x) dx &\leq \frac{C}{\lambda_i^{N+2\gamma} |d_{ij}|^{N+2\gamma}} \left(\frac{\lambda_j}{\lambda_i} \right)^{\frac{N-2\gamma}{2}} \int_0^{\lambda_i |d_{ij}|} \frac{r^{N-1} dr}{(1 + \frac{\lambda_j^2}{\lambda_i^2} r^2)^{\frac{N-2\gamma}{2}}} \\ &\leq \frac{C}{\lambda_i^{N+2\gamma} |d_{ij}|^{N+2\gamma}} \left(\frac{\lambda_i}{\lambda_j} \right)^{\frac{N+2\gamma}{2}} (\lambda_i \lambda_j |d_{ij}|^2)^\gamma = O(\varepsilon_{ij}^{\frac{N}{N-2\gamma}}). \end{aligned}$$

On the other hand, we have

$$S = \int_{\mathbb{R}^N} \delta_i^{\frac{N}{N-2\gamma}} \delta_j^{\frac{N}{N-2\gamma}} dx = C_0^{\frac{2N}{N-2\gamma}} \int_{\mathbb{R}^N} \frac{(\lambda_i \lambda_j)^{N/2}}{(1 + \lambda_i^2 |x - x_i|^2)^{\frac{N}{2}} (1 + \lambda_j^2 |x - x_j|^2)^{\frac{N}{2}}} dx$$

Let

$$a_{ij} = \frac{x_j - x_i}{2}, \quad z = x - \frac{x_i + x_j}{2}.$$

Then we have

$$S = C_0^{\frac{2N}{N-2\gamma}} \frac{1}{(\lambda_i \lambda_j)^{N/2}} \int_{\mathbb{R}^N} \frac{1}{\left(\frac{1}{\lambda_i^2} + |z + a_{ij}|^2 \right)^{\frac{N}{2}} \left(\frac{1}{\lambda_j^2} + |z - a_{ij}|^2 \right)^{\frac{N}{2}}} dx.$$

Combining the above computations, we have the desired estimates. This completes the proof. \square

LEMMA C.2. *For any $(y, \lambda) \in D_\mu$ and $v \in E_{y, \lambda}^2$, we have, for some $\tau > 0$,*

$$\int_{\mathbb{R}^n} K \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 1} v dx = O \left(\sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j}} + |y^j - z^j|^{\beta_j} \right) + \varepsilon_{12}^{\frac{1}{2} + \tau} \right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}.$$

PROOF. First, we have

$$\begin{aligned} \int_{\mathbb{R}^n} K(y) U_{x,\lambda}^{2_\gamma^*-1} v dy &= O \left(\int_{\mathbb{R}^n} \left(|x - z_j|^{\beta_j} + \frac{|y|^{\beta_j}}{\lambda^{\beta_j}} \right) U^{2_\gamma^*-1} |v(\frac{y}{\lambda} + x)| \lambda^{\frac{2\gamma-n}{2}} dy \right) \\ &= O \left(|x - z_j|^{\beta_j} + \frac{1}{\lambda^{\beta_j}} \right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}. \end{aligned}$$

Then

$$\begin{aligned} &\int_{\mathbb{R}^n} K \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^*-1} v dx \\ &= \sum_{j=1}^2 \int_{\mathbb{R}^n} K \alpha_j^{2_\gamma^*-1} U_{y^j, \lambda_j}^{2_\gamma^*-1} v dx + O \left(\int_{\mathbb{R}^n} K(y) U_{y^1, \lambda_1}^{2_\gamma^*-1} U_{y^2, \lambda_2}^{2_\gamma^*-1} |v| dx \right) \\ &= \sum_{j=1}^2 \int_{\mathbb{R}^n} K \alpha_j^{2_\gamma^*-1} U_{y^j, \lambda_j}^{2_\gamma^*-1} v dx + O \left(\varepsilon_{12}^{\frac{1}{2}+\tau} \right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}. \end{aligned}$$

Finally,

$$\int_{\mathbb{R}^n} K \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^*-1} v dx = \sum_{j=1}^2 \int_{\mathbb{R}^n} K \alpha_j^{2_\gamma^*-1} U_{y^j, \lambda_j}^{2_\gamma^*-1} v dx + O(\varepsilon_{12}) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}.$$

□

LEMMA C.3. For any $(y, \lambda) \in D_\mu$ and $v \in E_{y,\lambda}^2$, when μ is small, we have, for some $\tau > 0$,

$$\int_{\mathbb{R}^n} \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^*-1} v dx = O \left(\varepsilon_{12}^{\frac{1}{2}+\tau} \right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}.$$

PROOF.

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^*-1} v dx \\ &= \sum_{j=1}^2 \int_{\mathbb{R}^n} \alpha_j^{2_\gamma^*-1} U_{y^j, \lambda_j}^{2_\gamma^*-1} v dx + O \left(\int_{\mathbb{R}^n} U_{y^1, \lambda_1}^{2_\gamma^*-1} U_{y^2, \lambda_2}^{2_\gamma^*-1} |v| dx \right) \\ &= \sum_{j=1}^2 \int_{\mathbb{R}^n} \alpha_j^{2_\gamma^*-1} U_{y^j, \lambda_j}^{2_\gamma^*-1} v dx + O \left(\varepsilon_{12}^{\frac{1}{2}+\tau} \right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)} \\ &= O \left(\varepsilon_{12}^{\frac{1}{2}+\tau} \right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}. \end{aligned}$$

$$\begin{aligned}
& \int_{\mathbb{R}^n} \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 1} v dx \\
&= \sum_{j=1}^2 \int_{\mathbb{R}^n} \alpha_j^{2_\gamma^* - 1} U_{y^j, \lambda_j}^{2_\gamma^* - 1} v dx + O(\varepsilon_{12}) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)} \\
&= O(\varepsilon_{12}) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}.
\end{aligned}$$

□

LEMMA C.4. *Suppose $(y, \lambda) \in D_\mu$, if μ is small, then we have*

$$\begin{aligned}
& \left\langle \sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j}, U_{y^k, \lambda_k} \right\rangle - \int_{\mathbb{R}^N} (1 + \varepsilon K) \left(\sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 1} U_{y^k, \lambda_k} \\
&= O \left(\varepsilon \left(\frac{1}{\lambda_j^{\beta_k}} + |y^j - z^j|^{\beta_k} \right) + \varepsilon_{12}^{\frac{1}{2} + \tau} \right).
\end{aligned}$$

PROOF.

$$\begin{aligned}
& \left\langle \sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j}, U_{y^k, \lambda_k} \right\rangle - \int_{\mathbb{R}^N} (1 + \varepsilon K) \left(\sum_{j=1}^2 \hat{\alpha}_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 1} U_{y^k, \lambda_k} \\
&= \hat{\alpha}_k \left(\int_{\mathbb{R}^N} U_{y^k, \lambda_k}^{2_\gamma^*} - \int_{\mathbb{R}^N} \frac{1 + \varepsilon K}{1 + \varepsilon K(z^k)} U_{y^k, \lambda_k}^{2_\gamma^*} \right) + O \left(\int_{\mathbb{R}^N} U_{y^1, \lambda_1}^\alpha U_{y^2, \lambda_2}^\beta \right) \\
&= \frac{1}{(1 + \varepsilon K)^{\frac{N+2\gamma}{4\gamma}}} \int_{\mathbb{R}^N} \varepsilon (K(z^k) - K(x)) U_{y^k, \lambda_k}^{2_\gamma^*} + O(\varepsilon_{12}^{\frac{1}{2} + \tau}) \\
&= O(\varepsilon) \int_{\mathbb{R}^N} |(K(z^k) - K(\frac{y}{\lambda_k} + y_k))| \frac{1}{(1 + |y|^2)^N} + O(\varepsilon_{12}^{\frac{1}{2} + \tau}) \\
&= O \left(\varepsilon \left(\frac{1}{\lambda_j^{\beta_k}} + |y^j - z^j|^{\beta_k} \right) + \varepsilon_{12}^{\frac{1}{2} + \tau} \right).
\end{aligned}$$

□

LEMMA C.5. *Suppose $(y, \lambda) \in D_\mu$, if μ and ε are small, then we have*

$$\begin{aligned}
& \langle U_{y^l, \lambda_l}, U_{y^k, \lambda_k} \rangle - (2_\gamma^* - 1) \int_{\mathbb{R}^N} (1 + \varepsilon K) \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} U_{y^l, \lambda_l} U_{y^k, \lambda_k} \\
&= \begin{cases} (1 - (2_\gamma^* - 1) \alpha_k^{2_\gamma^* - 2}) A + O(\varepsilon_{12}^r) + O(\varepsilon), & \text{if } k = l = 1, 2, \\ O(\varepsilon_{12}^r) + O(\varepsilon), & \text{if } k \neq l, k, l = 1, 2 \end{cases}
\end{aligned}$$

for some $r > 0$, $A > 0$.

PROOF.

$$\begin{aligned}
& \langle U_{y^l, \lambda_l}, U_{y^k, \lambda_k} \rangle - (2_\gamma^* - 1) \int_{\mathbb{R}^N} (1 + \varepsilon K) \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} U_{y^l, \lambda_l}, U_{y^k, \lambda_k} \\
&= \int_{\mathbb{R}^N} U_{y^l, \lambda_l}^{2_\gamma^* - 1} U_{y^k, \lambda_k} - (2_\gamma^* - 1) \alpha_l^{2_\gamma^* - 2} \int_{\mathbb{R}^N} U_{y^l, \lambda_l}^{2_\gamma^* - 1} U_{y^k, \lambda_k} \\
&\quad + O \left(\int_{\mathbb{R}^N} \left(U_{y^1, \lambda_1}^{2_\gamma^* - 2} U_{y^2, \lambda_2}^2 + U_{y^1, \lambda_1}^{2_\gamma^* - 1} U_{y^2, \lambda_2} + U_{y^2, \lambda_2}^{2_\gamma^* - 2} U_{y^1, \lambda_1}^2 + U_{y^2, \lambda_2}^{2_\gamma^* - 1} U_{y^1, \lambda_1} \right) \right) + O(\varepsilon) \\
&= (1 - (2_\gamma^* - 1) \alpha_l^{2_\gamma^* - 2}) \int_{\mathbb{R}^N} U_{y^l, \lambda_l}^{2_\gamma^* - 1} U_{y^k, \lambda_k} + O(\varepsilon_{12}^r) + O(\varepsilon).
\end{aligned}$$

Let $A = \int_{\mathbb{R}^N} U_{y^k, \lambda_k}^{2_\gamma^*} = \int_{\mathbb{R}^N} U_{0,1}^{2_\gamma^*}$. We get the conclusion. \square

LEMMA C.6. For any $(y, \lambda) \in D_\mu$ and $v \in E_{y, \lambda}^2$, if μ and ε are small, then

$$\begin{aligned}
& \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} U_{y^k, \lambda_k} v dx \\
&= O \left(\varepsilon \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j}} + |y^j - z^j|^{\beta_j} \right) + \varepsilon_{12}^{\frac{1}{2} + \tau} \right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}.
\end{aligned}$$

PROOF.

$$\begin{aligned}
& \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} U_{y^k, \lambda_k} v dx \\
&= \alpha_k^{2_\gamma^* - 2} \int_{\mathbb{R}^n} U_{y^k, \lambda_k}^{2_\gamma^* - 1} v dx + \alpha_k^{2_\gamma^* - 2} \varepsilon \int_{\mathbb{R}^n} K(x) U_{y^k, \lambda_k}^{2_\gamma^* - 1} v dx \\
&\quad + \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} - (\alpha_k U_{y^k, \lambda_k})^{2_\gamma^* - 2} \right) U_{y^k, \lambda_k} v dx \\
&= \varepsilon O \left(\int_{\mathbb{R}^n} |x - z^k|^{\beta_k} U_{y^k, \lambda_k}^{2_\gamma^* - 1} |v| dx \right) + O \left(\int_{\mathbb{R}^n} U_{y^1, \lambda_1}^\alpha U_{y^2, \lambda_2}^\beta |v| dx \right) \\
&= O \left(\varepsilon \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j}} + |y^j - z^j|^{\beta_j} \right) + \varepsilon_{12}^{\frac{1}{2} + \tau} \right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}.
\end{aligned}$$

\square

Similarly, we have

LEMMA C.7. *For any $(y, \lambda) \in D_\mu$ and $v \in E_{y, \lambda}^2$, if μ and ε are small, then*

$$\begin{aligned} & \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} \frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k} v dx \\ &= O \left(\frac{\varepsilon}{\lambda_k} \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j}} + |y^j - z^j|^{\beta_j} \right) + \frac{\varepsilon_{12}^{\frac{1}{2} + \tau}}{\lambda_k} \right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}. \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbb{R}^n} (1 + \varepsilon K) \left(\sum_{j=1}^2 \alpha_j U_{y^j, \lambda_j} \right)^{2_\gamma^* - 2} \frac{\partial U_{y^k, \lambda_k}}{\partial y_i^k} v dx \\ &= O \left(\lambda_k \varepsilon \sum_{j=1}^2 \left(\frac{1}{\lambda_j^{\beta_j}} + |y^j - z^j|^{\beta_j} \right) + \lambda_k \varepsilon_{12}^{\frac{1}{2} + \tau} \right) \|v\|_{\dot{H}^\gamma(\mathbb{R}^N)}. \end{aligned}$$

LEMMA C.8. *For any $(y, \lambda) \in D_\mu$, if μ is small, then*

$$\begin{aligned} \int_{\mathbb{R}^N} K U_{y^k, \lambda_k}^{2_\gamma^* - 1} \frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k} dx &= C_{N, \beta_k} \frac{1}{\lambda_k^{\beta_k + 1}} \sum_{i=1}^N a_i^k + O \left(\frac{1}{\lambda_k^{\beta_k}} |y^k - z^k| \right) \\ &+ O \left(\frac{1}{\lambda_k^{\beta_k + 1 + \sigma}} \right) + O \left(\frac{1}{\lambda_k} |y^k - z^k|^{\beta_k + \sigma} \right), \end{aligned}$$

where C_{N, β_k} is a positive constant depending on N and β_k only.

PROOF.

$$\begin{aligned} & \int_{\mathbb{R}^n} K U_{y^k, \lambda_k}^{2_\gamma^* - 1} \frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k} dx \\ &= \int_{|x - z^k| \leq r_0} \left(\sum_{i=1}^N a_i^k |x_i - z_i^k|^{\beta_k} + O(|x - z^k|^{\beta_k + \sigma}) \right) U_{y^k, \lambda_k}^{2_\gamma^* - 1} \frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k} dx + O \left(\frac{1}{\lambda_k^{2N}} \right) \\ &= \frac{C_0^{2_\gamma^*}}{\lambda_k^{\beta_k + 1}} \int_{\mathbb{R}^N} \frac{N - 2\gamma}{2} \sum_{i=1}^N a_i^k |x_i + \lambda_k(y_i^k - z_i^k)|^{\beta_k} \frac{1 - |x|^2}{(1 + |x|^2)^{N+1}} dx \\ &+ O \left(\frac{1}{\lambda_k^{\beta_k + 1 + \sigma}} \right) + O \left(\frac{1}{\lambda_k} |y^k - z^k|^{\beta_k + \sigma} \right) \\ &= \frac{C_0^{2_\gamma^*}}{\lambda_k^{\beta_k + 1}} \frac{N - 2\gamma}{2} \int_{\mathbb{R}^N} \sum_{i=1}^N a_i^k |x_i|^{\beta_k} \frac{1 - |x|^2}{(1 + |x|^2)^{N+1}} dx \\ &+ O \left(\frac{1}{\lambda_k^{\beta_k}} |y^k - z^k| \right) + O \left(\frac{1}{\lambda_k^{\beta_k + 1 + \sigma}} \right) + O \left(\frac{1}{\lambda_k} |y^k - z^k|^{\beta_k + \sigma} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{C_0^{2\gamma}}{\lambda_k^{\beta_k+1}} \frac{N-2\gamma}{2N} \sum_{i=1}^N a_i^k \int_{\mathbb{R}^N} |x|^{\beta_k} \frac{1-|x|^2}{(1+|x|^2)^{N+1}} dx \\
&\quad + O\left(\frac{1}{\lambda_k^{\beta_k}} |y^k - z^k|\right) + O\left(\frac{1}{\lambda_k^{\beta_k+1+\sigma}}\right) + O\left(\frac{1}{\lambda_k} |y^k - z^k|^{\beta_k+\sigma}\right) \\
&= \frac{C_0^{2\gamma}}{\lambda_k^{\beta_k+1}} \frac{N-2\gamma}{2N} \sum_{i=1}^N a_i^k \frac{2n-2\beta_k-4}{2n-\beta_k-3} \int_{\mathbb{R}^N} |x|^{\beta_k} \frac{1}{(1+|x|^2)^{N+1}} dx \\
&\quad + O\left(\frac{1}{\lambda_k^{\beta_k}} |y^k - z^k|\right) + O\left(\frac{1}{\lambda_k^{\beta_k+1+\sigma}}\right) + O\left(\frac{1}{\lambda_k} |y^k - z^k|^{\beta_k+\sigma}\right) \\
&= C_0^{2\gamma} \frac{N-2\gamma}{2N} \frac{2n-2\beta_k-4}{2n-\beta_k-3} \frac{1}{\lambda_k^{\beta_k+1}} \int_{\mathbb{R}^N} |x|^{\beta_k} \frac{1}{(1+|x|^2)^{N+1}} dx \sum_{i=1}^N a_i^k \\
&\quad + O\left(\frac{1}{\lambda_k^{\beta_k}} |y^k - z^k|\right) + O\left(\frac{1}{\lambda_k^{\beta_k+1+\sigma}}\right) + O\left(\frac{1}{\lambda_k} |y^k - z^k|^{\beta_k+\sigma}\right),
\end{aligned}$$

□

LEMMA C.9. For any $(y, \lambda) \in D_\mu$, if μ is small, then

$$\begin{aligned}
\int_{\mathbb{R}^N} KU_{y^k, \lambda_k}^{2^*-1} \frac{\partial U_{y^k, \lambda_k}}{\partial y_i^k} dx &= D_{N, \beta_k} \frac{1}{\lambda_k^{\beta_k-1}} a_i^k (y_i^k - z_i^k) + O\left(\frac{1}{\lambda_k^{\beta_k-1}} \lambda_k^2 |y^k - z^k|^2\right) \\
&\quad + O\left(\frac{1}{\lambda_k^{\beta_k-1+\sigma}}\right) + O(\lambda_k |y^k - z^k|^{\beta_k+\sigma}),
\end{aligned}$$

where D_{N, β_k} is a positive constant depending on N and β_k only.

PROOF.

$$\begin{aligned}
\int_{\mathbb{R}^N} KU_{y^k, \lambda_k}^{2^*-1} \frac{\partial U_{y^k, \lambda_k}}{\partial y_i^k} dx &= (N-2\gamma) \int_{\mathbb{R}^N} KU_{y^k, \lambda_k}^{2\gamma} \frac{\lambda_k^2 (x_i - y_i^k)}{1 + \lambda_k^2 |x - y^k|^2} dx \\
&= (N-2\gamma) \int_{|x-z^k| \leq r_0} \left(\sum_{h=1}^N a_h^k |x_h - z_h^k|^{\beta_k} + O(|x - z^k|^{\beta_k+\sigma}) \right) U_{y^k, \lambda_k}^{2\gamma} \frac{\lambda_k^2 (x_i - y_i^k)}{1 + \lambda_k^2 |x - y^k|^2} dx \\
&\quad + O\left(\frac{1}{\lambda_k^{N-1}}\right) \\
&= (N-2\gamma) \lambda_k \int_{\mathbb{R}^N} \sum_{h=1}^N a_h^k |x_h - z_h^k|^{\beta_k} U_{y^k, \lambda_k}^{2\gamma} \frac{\lambda_k (x_i - y_i^k)}{1 + \lambda_k^2 |x - y^k|^2} dx \\
&\quad + O\left(\frac{1}{\lambda_k^{\beta_k-1+\sigma}} + \lambda_k |y^k - z^k|^{\beta_k+\sigma}\right)
\end{aligned}$$

$$\begin{aligned}
&= (N - 2\gamma) \frac{1}{\lambda_k^{\beta_k-1}} \int_{\mathbb{R}^N} \sum_{h=1}^N a_h^k (|x_h|^{\beta_k} + \beta_k |x_h|^{\beta_k-2} x_h \lambda_k (y_h^k - z_h^k)) U_{0,1}^{2\gamma} \frac{x_i}{1 + |x|^2} dx \\
&\quad + O \left(\frac{1}{\lambda_k^{\beta_k-1}} \lambda_k^2 |y^k - z^k|^2 \right) + O \left(\frac{1}{\lambda_k^{\beta_k-1+\sigma}} + \lambda_k |y^k - z^k|^{\beta_k+\sigma} \right) \\
&= C_0^{2\gamma} \frac{(N - 2\gamma)}{N} \frac{\beta_k}{\lambda_k^{\beta_k-1}} a_i^k \int_{\mathbb{R}^N} \frac{|x|^{\beta_k}}{(1 + |x|^2)^{N+1}} dx (y_h^k - z_h^k) \\
&\quad + O \left(\frac{1}{\lambda_k^{\beta_k-1}} \lambda_k^2 |y^k - z^k|^2 \right) + O \left(\frac{1}{\lambda_k^{\beta_k-1+\sigma}} + \lambda_k |y^k - z^k|^{\beta_k+\sigma} \right).
\end{aligned}$$

□

Similarly, we have

LEMMA C.10. *For any $(y, \lambda) \in D_\mu$, if μ is small and $k \neq l$, then*

$$\int_{\mathbb{R}^N} U_{y^k, \lambda_k}^{2\gamma-2} \frac{\partial U_{y^k, \lambda_k}}{\partial \lambda_k} U_{y^l, \lambda_l} dx = \frac{1}{2_\gamma^* - 1} C_0 \frac{\varepsilon_{12}}{\lambda_k |z^1 - z^2|^{N-2\gamma}} + O \left(\frac{\varepsilon_{12}^{\frac{N}{N-2\gamma}}}{\lambda_k} \right).$$

and

LEMMA C.11. *For any $(y, \lambda) \in D_\mu$, if μ is small and $k \neq l$, then*

$$\int_{\mathbb{R}^N} U_{y^k, \lambda_k}^{2\gamma-2} \frac{\partial U_{y^k, \lambda_k}}{\partial y_i^k} U_{y^l, \lambda_l} dx = C_1 \lambda_1 \lambda_2 (y_i^k - y_i^l) \varepsilon_{12}^{\frac{N}{N-2\gamma}} + O \left(\varepsilon_{12}^{\frac{N-1}{N-2\gamma}} \right).$$

Appendix D.

The integration by parts formula for the fractional Laplacian is frequently used in this paper. We give its proof here.

LEMMA D.1. *If f and g belongs to the space $\dot{H}^{2s}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then it holds that*

$$\int_{\mathbb{R}^N} (-\Delta)^s f(x) g(x) dx = \int_{\mathbb{R}^N} (-\Delta)^s g(x) f(x) dx$$

PROOF.

$$\begin{aligned}
\int_{\mathbb{R}^N} (-\Delta)^s f(x) g(x) dx &= \int_{\mathbb{R}^N} g(x) \int_{\mathbb{R}^N} \frac{f(x) - f(y)}{|x - y|^{N+2s}} dy dx \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(x)(f(x) - f(y))}{|x - y|^{N+2s}} dy dx \\
&= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(y)(f(y) - f(x))}{|x - y|^{N+2s}} dy dx \\
&= - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{g(y)(f(x) - f(y))}{|x - y|^{N+2s}} dy dx.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^N} (-\Delta)^s f(x) g(x) dx &= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(g(x) - g(y))(f(x) - f(y))}{|x - y|^{N+2s}} dy dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (-\Delta)^s g(x) f(x) dx - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(g(x) - g(y))f(y)}{|x - y|^{N+2s}} dy dx \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (-\Delta)^s g(x) f(x) dx + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(g(x) - g(y))f(x)}{|x - y|^{N+2s}} dy dx \\
&= \int_{\mathbb{R}^N} (-\Delta)^s g(x) f(x) dx.
\end{aligned}$$

□

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SCHOOL OF MATHEMATICAL SCIENCE, NANKAI UNIVERSITY, TIANJIN 300071, P.R. CHINA
E-mail address: cyh1801@163.com

SCHOOL OF SCIENCE, TIANJIN UNIVERSITY, TIANJIN 300072, P. R. CHINA.
E-mail address: zhengyq@tju.edu.cn